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OPTIMAL CONTROL OF LINEAR TIME DELAY SYSTEMS

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ABSTRACT

A method is presented by which an optimal control may be obtained for a linear time varying system with time delay. The performance criterion is quadratic with a fixed finite upper limit. The results are a set of partial differential equations with boundary conditions, whose solution yields an optimal feedback control. The discussion includes possible methods for the solution of the partial differential equations.

I. INTRODUCTION

Many physical systems are best modeled by the use of time delay equations [1]; that is, a differential equation of the form

$$\dot{x}(t) = F(t, x(t), x(t-h_1), \ldots, x(t-h_n), u(t))$$

In some systems of this type it is desirable to select certain control parameters to minimize a performance criterion modeled by a cost functional of the form

$$J(u) = G(x(T), T) + \int_{t_o}^{T} g(t, x(t), u(t))dt.$$

Kharatishvili [2] has approached this problem by extending Pontryagin's Maximum Principle to time delay systems. The actual solution involves a two point boundary value problem in which advances and delays are present. In addition, this solution does not yield a feedback controller. Time optimal control of delay systems has been considered by Oguztoreli [3]. He has obtained several results concerning "bang-bang" controls which parallel those of LaSalle [4] for non-delay systems. For a time invariant system with an infinite upper limit Krasovskii [5] has developed a closed loop controller involving an integral over the delay period.

For a linear time varying system with a single delay Krasovskii [6] states that if a quadratic cost functional is used, the optimal cost functional must be of the form

$$x'(t_o)p_1(t_o)x(t_o) + x'(t_o)\int_{-h}^{o} p_2(t_o, s)x(t_o+s)ds$$
$$+ \int_{-h}^{o}\int_{-h}^{o} x'(t_o+s)p_3(t_o, r, s)\ x(t_o+r)dr\,ds,$$

where h is the delay. These results, however, are not proven and no information is given on how to obtain $p_1(t_o)$, $p_2(t_o, s)$ and $p_3(t_o, r, s)$.

This paper uses Krasovskii's [6] results to develop a set of partial differential equations which may be solved for the optimal feedback controller of a linear time varying system with delay. The cost functional is quadratic and has a finite upper limit. Although the results obtained are for systems with a single delay, they are easily extended to multiple delay systems. The partial differential equations obtained are discussed and some techniques for solutions are mentioned.

Under the restriction that the initial time t_o of the cost functional is within a delay period of the final time T, the problem is also solved by the use of a known technique and the results compared.

II. NOTATION AND PROBLEM STATEMENT

Consider a system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) + D(t)u(t) \tag{1}$$

where x(t) is an n-vector; u(t) is an m-vector, the control; A(t), B(t) and D(t) are nxn, nxn and nxm matrices respectively, which are continuous in t, the time; h is a positive scalar constant, the delay. The following notation will be used:

(i) The dot denotes the time derivative.

(ii) ' denotes the transpose.

(iii) $C[-h, 0]$ and $C^1[-h, 0]$ will denote the space of functions continuous on $[-h, 0]$ and the space of differentiable functions on $[-h, 0]$ respectively.

(iv) $x_\tau(s) \triangleq x(\tau+s) = \varphi(s)$, $s\varepsilon[-h, 0]$ where $\varphi\varepsilon\ C[-h, 0]$, and therefore the convention $x_\tau\ \varepsilon\ C[-h, 0]$.

(v) x^u denotes a solution of (1) for $t\varepsilon[\tau, T]$, $x_\tau = \varphi$ and $\tau\varepsilon[t_o, T]$.

(vi) Let $V(t,\varphi):[t_o, T] \times C[-h, 0] \rightarrow R^1$ be a continuous functional, define

$$\frac{dV(t,\varphi)}{dt}\Big|_{u(t)} \triangleq \lim_{\Delta t \to o} \left[\frac{V(t+\Delta t, x^u_{t+\Delta t}) - V(t,\varphi)}{\Delta t}\right]$$

where $x^u_{t+\Delta t}$ represents the solution of (1) and φ is the initial function at time t.

(v) R^1 denotes the reals.

A control is desired which will minimize the functional

$$J(t_o, x_{t_o}, u) = x'(T)Fx(T)\Big|_{x=x^u} + \int_{t_o}^{T} \{x'(t)Q(t)x(t) + u(t)R(t)u(t)\}_{x=x^u} dt \quad (2)$$

subject to (1). Such a control will be called optimal. In (2), F is a symmetric, positive semi-definite nxn matrix; Q(t) is a continuous, symmetric, positive semi-definite nxn matrix; R(t) is a continuous, symmetric positive definite mxm matrix; $t_o \leq T-h$.

Preliminary Results

The following lemma establishes a sufficient condition for a control u to be optimal.

Lemma 1. If it is possible to find a continuous functional $V(t,\varphi):[t_o, T] \times C[-h, 0] \rightarrow R^1$, and a function u0(t) such that

(i) $V(T,\varphi) = \varphi'(0)F\varphi(0)$

and in view of Equation (1) with φ as an initial condition.

$$\text{(ii)} \quad \frac{dV(t,\varphi)}{dt}\Big|_{u0(t)} + \varphi'(0)Q(t)\varphi(0) + u0'(t)R(t)u0'(t) = 0 \qquad (3)$$

$$\text{(iii)} \quad \frac{dV(t,\varphi)}{dt}\Big|_{u(t)} + \varphi'(0)Q(t)\varphi(0) + u'(t)R(t)u(t) \geq$$

$$\frac{dV(t,\varphi)}{dt}\Big|_{u0(t)} + \varphi'(0)Q(t)\varphi(0)+u0'(t)R(t)u0(t) \tag{4}$$

$\forall t\in[t_o, T]$ and $\forall\varphi\in C[-h,0]$.

Then

$$V(\tau,\varphi) = J(\tau,\varphi,u0)$$

and

$$J(\tau,\varphi,u0) = V(\tau,\varphi) \leq J(\tau,\varphi,u),\ \forall\tau\in[t_o,T] \text{ and } \forall\varphi\in C[-h,0].$$

Proof. In the evaluation of Equation (3) along a trajectory of (1), $\varphi(0)$ becomes $x_t^u(0)$. Therefore, the integration of Equation (3) along a trajectory of (1) with $u(t) = u0(t)$ yields

$$\int_\tau^T \left\{\frac{dV(t,x_t)}{dt} + x_t'(0)Q(t)x_t(0)+u0'(t)R(t)u0(t)\right\}_{x=x^{uo}} dt = 0$$

but $x_t(0) = x(t)$ so

$$\int_\tau^T \left\{\frac{dV(t,x_t)}{dt} + x'(t)Q(t)x(t)+u0'(t)R(t)u0(t)\right\}_{x=x^{uo}} dt = 0$$

where

$x_\tau = \varphi$, an arbitrary element of $C[-h,0]$. Then

$$-V(\tau,\varphi)+V(T,x_T)\Big|_{x=x^{uo}} = -\int_\tau^T \{x'(t)Q(t)x(t)+u0'(t)R(t)u0(t)\}_{x=x^{uo}}\, dt.$$

From Condition (i) of the theorem it is seen that

$$-V(\tau,\varphi)+x'(T)Fx(T)\Big|_{x=x^{uo}} = -\int_\tau^T \{x'(t)Q(t)x(t)+u0'(t)R(t)u0(t)\}_{x=x^{uo}}\, dt$$

and hence

$$J(\tau,\varphi,u0) = V(\tau,\varphi)$$

$$= x'(T)Fx(T)\Big|_{x=x^{uo}} + \int_\tau^T \{x'(t)Q(t)x(t)+u0'(t)R(t)u0(t)\}_{x=x^{uo}}\, dt$$

$\forall\tau\in[t_o,T]$ and $\forall\varphi\in C[-h,0]$.

Integration of Equation (4) from τ to T yields

$$\int_{\tau}^{T} \left\{ \frac{dV(t,x_t)}{dt} + x'(t)Q(t)x(t)+u'(t)R(t)u(t)\right\}_{x=x^u} dt \geq 0$$

where $x_\tau = \varphi$. Then

$$-V(\tau,\varphi)+x'(T)Fx(T)\Big|_{x=x^u} + \int_{\tau}^{T} \{x'(t)Q(t)x(t)+u'(t)R(t)u(t)\}_{x=x^u} dt \geq 0$$

which implies

$$J(\tau,\varphi,u0) \leq J(\tau,\varphi,u), \ \forall\tau\in[t_o,T] \text{ and } \forall\varphi\in C[-h,0].$$

Q.E.D.

III. AN OPTIMAL FEEDBACK CONTROL

Lemma 1 will now be used to obtain sufficient conditions for an optimal control in feedback form. From this point on x_t will be used to indicate the part of the solution x^u from t-h to t. It should be noted that if t is the initial time, $x_t = \varphi$, an arbitrary element of C[-h, 0].

A form for $V(t,x_t)$ will be assumed and the unknown variables in the form chosen so as to satisfy conditions (i), (ii) and (iii) of Lemma 1. It will be shown later that if $V(t,x_t)$ is of the form

$$V(t,x_t) = x'(t)y_1(t)x(t)+x'(t)\int_{-h}^{0} y_2(t,s)x(t+s)ds$$

$$+\int_{-h}^{0}\int_{-h}^{0} x'(t+s)y_3(t,r,s)x(t+r)drds, \tag{5}$$

then the following will be satisfied for all r and s contained in [-h, 0]:

$$y_2(t,s) = 0 \text{ for } T-h < t+s \tag{6}$$

and

$$y_3(t,r,s) = 0 \text{ for } T-h < t+s \text{ or } T-h < t+r \tag{7}$$

and for $F \neq 0$, $y_2(t,s)$ is discontinuous at $s+t = T-h$ and $y_3(t,r,s)$ is discontinuous at

$$s+t = T-h \text{ and } r+t \leq T-h \text{ or } r+t = T-h \text{ and } s+t \leq T-h.$$

Because of the above two separate cases will be considered.

Case 1 ($t < T-h$)

Define

$$V_1(t,x_t) \triangleq x'(t)p_1(t)x(t) + 2x'(t)\int_{-h}^{0} p_2(t,s)x(t+s)ds$$

$$+\int_{-h}^{0}\int_{-h}^{0} x'(t+s)p_3(t,r,s)x(t+s)ds \tag{8}$$

where $p_1(t)$, $p_2(t,s)$, $p_3(t,r,s)$ are nxn matrices. It is assumed without loss of generality [see Appendix I] that $p_1(t)$ is symmetric and

$$p_3(t,r,s) = p_3'(t,s,r).$$

$p_2(t,s)$ and $p_3(t,r,s)$ are assumed differentiable with respect to t, r and s in the region $t < T-h$.

In order to evaluate $\frac{dV_1(t,x_t)}{dt}$ it will be necessary to differentiate $x(t+s)$ with respect to t for $s \in [-h,0]$, therefore assume $x_t \in C^1[-h,0]$, the space of differentiable functions. The differentiation of Equation (8) with respect to time yields

$$\frac{dV_1(t,x_t)}{dt} = \dot{x}'(t)p_1(t)x(t)+x'(t)\dot{p}_1(t)x(t)$$

$$+ x'(t)p_1(t)\dot{x}(t)+2\dot{x}'(t)\int_{-h}^{0} p_2(t,s)x(t+s)ds$$

$$+2x'(t)\int_{-h}^{0}\frac{\partial p_2(t,s)}{\partial t}x(t+s)ds + 2x'(t)\int_{-h}^{0} p_2(t,s)\dot{x}(t+s)ds$$

$$+\int_{-h}^{0}\int_{-h}^{0}\dot{x}'(t+s)p_3(t,r,s)x(t+r)drds$$

$$+\int_{-h}^{0}\int_{-h}^{0} x'(t+s)\,\frac{\partial p_3(t,r,s)}{\partial t}\,x(t+r)drds$$

$$+\int_{-h}^{0}\int_{-h}^{0} x'(t+s)p_3(t,r,s)\dot{x}(t+r)drds \tag{9}$$

Using Equation (9), $\frac{dV_1(t,x_t)}{dt}$ is now evaluated along the trajectories of (1). Noting that

$$\frac{dx(t+s)}{dt}=\frac{dx(t+s)}{ds}$$

and that $p_2(t,s)$ and $p_3(t,r,s)$ are differentiable, integration by parts yields

$$\frac{dV_1(t,x_t)}{dt}\Big|_{x=x^u}=2x'(t)A'(t)p_1(t)x(t)+2x'(t-h)B'(t)p_1(t)x(t)$$

$$+2u'(t)D(t)p_1(t)x(t)+x'(t)\dot{p}_1(t)x(t)+2x'(t)A'(t)\int_{-h}^{0}p_2(t,s)x(t+s)ds$$

$$+2x'(t-h)B'(t)\int_{-h}^{0}p_2(t,s)x(t+s)ds+2u'(t)D'(t)\int_{-h}^{0}p_2(t,s)x(t+s)ds$$

$$+2x(t)\int_{-h}^{0}\frac{\partial p_2(t,s)}{\partial t}x(t+s)ds+2x'(t)p_2(t,s)x(t+s)\Big|_{s=-h}^{s=o}$$

$$+\int_{-h}^{0}\int_{-h}^{0}x'(t+s)\,\frac{\partial p_3(t,r,s)}{\partial t}\,x(t+r)drds-2x'(t)\int_{-h}^{0}\frac{\partial p_2(t,s)}{\partial s}x(t+s)ds$$

$$+\int_{-h}^{0}[x'(t+s)p_3(t,r,s)x(t+r)\Big|_{r=-h}^{r=o}-\int_{-h}^{0}x'(t+s)\frac{\partial p_3(t,r,s)}{\partial r}x(t+r)dr]ds$$

$$+\int_{-h}^{0}[x'(t+s)p_3(t,r,s)x(t+r)\Big|_{s=-h}^{s=o}-\int_{-h}^{0}x'(t+s)\frac{\partial p_3(t,r,s)}{\partial s}x(t+r)ds]dr.$$

Adding $x'(t)Q(t)x(t)+u'(t)R(t)u(t)$ to both sides and grouping terms yields

$$\frac{dV_1(t,x_t)}{dt}\Big|_{x=x^u} + x'(t)Q(t)x(t)+u'(t)R(t)u(t) = x'(t)[2A'(t)p_1(t)+Q(t)+\dot{p}_1(t)$$

$$+2p_2(t,o)]x(t)+x'(t-h)[2B'(t)p_1(t)-2p_2'(t,-h)]x(t)+x'(t)\int_{-h}^{0}[-2\frac{\partial p_2(t,s)}{\partial s}$$

$$+2\frac{\partial p_2(t,s)}{\partial t}+2A'(t)p_2(t,s)+2p_3'(t,0,s)]x(t+s)ds+x'(t-h)\int_{-h}^{0}[2B'(t)p_2(t,s)$$

$$-2p_3'(t,-h,s)]x(t+s)ds+\int_{-h}^{0}\int_{-h}^{0}x'(t+s)[\frac{\partial p_3(t,r,s)}{\partial t}$$

$$-\frac{\partial p_3(t,r,s)}{\partial r}-\frac{\partial p_3(t,r,s)}{\partial s}]x(t+r)drds+2u'(t)D'(t)p_1(t)x(t)$$

$$+2u'(t)D'(t)\int_{-h}^{0}p_2(t,s)x(t+s)ds+u'(t)R(t)u(t) \qquad (10)$$

Now a u1 will be found such that condition (iii) of Lemma 1 is satisfied.

From Equation (10) it is seen that

$$\frac{\partial\left\{\frac{dV_1(t,x_t)}{dt}\Big|_{x=x^u} + x'(t)Q(t)x(t)+u'(t)R(t)u(t)\right\}}{\partial u(t)} = 0$$

if and only if

$$u(t) = -R^{-1}(t)D'(t)p_1(t)x(t)-R^{-1}(t)D'(t)\int_{-h}^{0}p_2(t,s)x(t+s)ds \qquad (11)$$

Since $\dfrac{\partial^2\left\{\frac{dV_1(t,x_t)}{dt}\Big|_{x=x^u} + x'(t)Q(t)x(t)+u'(t)R(t)u(t)\right\}}{\partial u(t)^2} = R(t)$ and

$$\frac{\partial^n\left\{\frac{dV_1(t,x_t)}{dt}\Big|_{x=x^u} + x'(t)Q(t)x(t)+u'(t)R(t)u(t)\right\}}{\partial u^n} = 0,\ n > 2,$$

u(t) as expressed in Equation (11) is a global minimum of

$$\frac{dV_1(t,x_t)}{dt}\Big|_{x=x^u} + x'(t)Q(t)x(t)+u'(t)R(t)u(t)$$

and condition (iii) of Lemma 1 is satisfied by

$$u1(t) = -R^{-1}(t)D'(t)p_1(t)x(t) - R^{-1}(t)D'(t)\int_{-h}^{0} p_2(t,s)x(t+s)ds. \qquad (12)$$

u(t) appears only in the last three terms (L.T.T.) of Equation (10), which on substitution of u1(t) for u(t) yields

$$\begin{aligned}
\text{L.T.T.} = &-2x'(t)p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)x(t) \\
&-\int_{-h}^{0} 2x'(t+s)p_2'(t,s)D(t)R^{-1}(t)D'(t)p_1(t)x(t)ds \\
&-2x(t)p_1(t)D(t)R^{-1}(t)D'(t)\int_{-h}^{0} p_2(t,s)x(t+s)ds \\
&-2\int_{-h}^{0} x'(t+s)p_2'(t,s)ds\, D(t)R^{-1}(t)D'(t)\int_{-h}^{0} p_2(t,s)x(t+s)ds \\
&+x'(t)p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)x(t) \\
&\int_{-h}^{0}\int_{-h}^{0} x'(t+s)p_2'(t,s)D(t)R^{-1}(t)D'(t)p_2(t,r)x(t+r)drds \\
&+x'(t)p_1(t)D(t)R^{-1}(t)D'(t)\int_{-h}^{0} p_2(t,s)x(t+s)ds \\
&+\int_{-h}^{0} x'(t+s)p_2'(t,s)ds\, D(t)R^{-1}(t)D'(t)p_1(t)x(t),
\end{aligned}$$

which reduces to

$$\begin{aligned}
\text{L.T.T.} = &-x'(t)p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)x(t) \\
&-2x'(t)\int_{-h}^{0} p_1(t)D(t)R^{-1}(t)D'(t)p_2(t,s)x(t+s)ds \\
&-\int_{-h}^{0}\int_{-h}^{0} x'(t+s)p_2'(t,s)D(t)R^{-1}(t)D'(t)p_2(t,r)x(t+r)drds \qquad (14)
\end{aligned}$$

Utilizing Equation (14), Equation (10) becomes

$$\begin{aligned}
\left.\frac{dV_1(t,x_t)}{dt}\right|_{x=x^u} &+x'(t)D(t)x(t)+u'(t)R(t)u(t)=x'(t)[2p(t)p_1(t)+Q(t)+\dot{p}_1(t) \\
&+2p_2(t,0)-p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)]x(t)+2x'(t-h)[B'(t)p_1(t)-p_2'(t,-h]x(t) \\
&+x'(t)\int_{-h}^{0}[-2\frac{\partial p_2(t,s)}{\partial s}+2\frac{\partial p_2(t,s)}{\partial t}+2A'(t)p_2(t,s)+2p_3(t,0,s)
\end{aligned}$$

$$-2p_1(t)D(t)R^{-1}(t)D'(t)p_2(t,s)]x(t+s)ds$$

$$+x'(t-h)\int_{-h}^{0}[2B'(t)p_2(t,s)-2p_3(t,-h,s)]x(t+s)ds$$

$$+\int_{-h}^{0}\int_{-h}^{0}x'(t+s)[\frac{\partial p_3(t,r,s)}{\partial t}-\frac{\partial p_3(t,r,s)}{\partial s}-\frac{\partial p_3(t,r,s)}{\partial r}$$

$$-p_2'(t,s)D(t)R^{-1}(t)D'(t)p_2(t,r)]x(t+r)drds \tag{15}$$

In order to satisfy condition (ii) of Lemma 1, Equation (15) is equated to zero. This must hold for all $t_0 < t < T-h$ where x_t is a completely arbitrary element of $C^1[-h,0]$. A necessary and sufficient condition for this equality is that the following equations hold for $t < T-h$:

$$\dot{p}_1(t)-p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)+p_2(t,0)+p_2'(t,0)+A'(t)p_1(t)$$
$$+p_1(t)A(t)+Q(t) = 0 \tag{16}$$

$$\frac{\partial p_2(t,s)}{\partial t}-\frac{\partial p_2(t,s)}{\partial s}+A'(t)p_2(t,s)+p_3'(t,0,s)-p_1(t)D(t)R^{-1}(t)D'(t)p_2(t,s)=0 \tag{17}$$

$$\frac{\partial p_3(t,r,s)}{\partial t}-\frac{\partial p_3(t,r,s)}{\partial s}-\frac{\partial p_3(t,r,s)}{\partial r}-p_2'(t,s)D(t)R^{-1}(t)D'(t)p_2(t,r)=0 \tag{18}$$

$$B'(t)p_1(t) = p_2'(t,-h) \tag{19}$$

and

$$B'(t)p_2(t,s) = p_3(t,-h,s) \tag{20}$$

where $-h \leq s \leq 0$, $-h \leq r \leq 0$.

The above equations are clearly sufficient. It remains to be shown that they are necessary.

Letting $x(t) = 0$ and $x(t-h) = 0$ and equating (15) to zero yields

$$\int_{-h}^{0}\int_{-h}^{0}x'(t+s)[\frac{\partial p_3(t,r,s)}{\partial t}-\frac{\partial p_3(t,r,s)}{\partial s}-\frac{\partial p_3(t,r,s)}{\partial r}$$
$$-p_2'(t,s)D(t)R^{-1}(t)D'(t)p_2(t,r)]x(t+r)drds = 0$$

Since the integrand is assumed continuous Corollary 1 of Appendix 2 states that Equation (18) is necessary.

Letting $x(t) = 0$ and equating (15) to zero yields

$$x(t-h)\int_{-h}^{0} [2B^T(t)p_2(t,s)-2p_3(t,-h,s)]x(t+s)ds = 0.$$

From Corollary 2 Appendix 2 Equation (20) is seen to be necessary.

Now let $x(t-h) = 0$ and equate (15) to zero. It is seen that

$$x'(t)[2A(t)p_1(t)+Q(t)+\dot{p}_1(t)+2p_2(t,0)-p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)]x(t)$$

$$+x'(t)\int_{-h}^{0}[-2\frac{\partial p_2(t,s)}{\partial s}+2\frac{\partial p_2(t,s)}{\partial t}+2A'(t)p_2(t,s)+2p_3(t,0,s)$$

$$-2p_1(t)D(t)R^{-1}(t)D'(t)p_2(t,s)]x(t+s)ds = 0.$$

It is apparent that for this to be true

$$x'(t)[2A(t)p_1(t)+Q(t)+\dot{p}_1(t)+2p_2(t,0)-p_1(t)D(t)R^{-1}(t)D'(t)p_1(t)]x(t)=0 \tag{21}$$

and

$$x'(t)\int_{-h}^{0}[-2\frac{\partial p_2(t,s)}{\partial s}+2\frac{\partial p_2(t,s)}{\partial t}+2A'(t)p_2(t,s)+2p_3(t,0,s)$$

$$-2p_1(t)D(t)R^{-1}(t)D'(t)p_2(t,s)]x(t+s)ds = 0 \tag{22}$$

From Corollary 2 Appendix 2 for (22) to be satisfied Equation (17) is necessary and for (21) to be satisfied it is necessary that (16) be satisfied. Now equating (15) to zero yields

$$2x'(t-h)[B'(t)p_1(t)-p_2'(t,-h)]x(t) = 0 \tag{23}$$

For this to be true (19) is necessary. Thus (16)-(20) are seen to be necessary for expression (15) to be zero.

Case II ($t \geq T-h$)

Define

$$V_2(t,x_t) = x'(t)w_1(t)x(t)+2x'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds$$

$$+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t} x'(t+s)w_3(t,r,s)x(t+r)drds \tag{24}$$

where $w_1(t)$, $w_2(t,s)$ and $w_3(t,r,s)$ are nxn matrices,

$w_2(t,s)$ is assumed differentiable for $t \geq T-h$ and $-h \leq s \leq T-h-t$,

$w_3(t,r,s)$ is assumed differentiable for $t \geq T-h$ and $-h \leq s \leq T-h-t$ and $-h \leq r \leq T-h-t$.

This form is chosen since Equations (6) and (7) imply that

$$\int_{T-h-t}^{0} w_2(t,s)x(t+s)ds = 0$$

and

$$\int_{T-h-t}^{0} x'(t+s)w_3(t,r,s)x(t+r)dr = 0.$$

It is assumed without loss of generality that $w_1(t) = w_1'(t)$ and $w_3(t,r,s) = w_3'(t,s,r)$.

Differentiation of Equation (24) with respect to time and noting that

$$\frac{dx(t+s)}{dt} = \frac{dx(t+s)}{ds}$$

yields

$$\frac{dV_2(t,x_t)}{dt} = \dot{x}'(t)w_1(t)x(t)+x'(t)\dot{w}_1(t)x(t)+x'(t)w_1(t)\dot{x}(t)$$

$$+2\dot{x}'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds+2x'(t)\int_{-h}^{T-h-t}\frac{\partial w_2(t,s)}{\partial t}x(t+s)ds$$

$$+2x'(t)\int_{-h}^{T-h-t} w_2(t,s)\frac{dx(t+s)}{ds}ds-2x'(t)w_2(t,T-h-t)x(T-h)$$

$$+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t} \frac{dx'(t+s)}{ds} w_3(t,r,s)x(t+r)drds$$

$$+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t} x'(t+s)w_3(t,r,s) \frac{dx(t+r)}{dr} drds$$

$$+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t} x'(t+s) \frac{\partial w_3(t,r,s)}{\partial t} x(t+r)drds$$

$$-\int_{-h}^{T-h-t} x'(t+s)w_3(t,T-h-t,s)x(T-h)ds$$

$$-\int_{-h}^{T-h-t} x'(T-h)w_3(t,r,T-h-t)x(t+r)dr. \tag{25}$$

Using Equation (25), $\frac{dV_2(t,x_t)}{dt}$ is now evaluated along the trajectories of (1). Since $w_2(t,s)$ and $w_3(t,r,s)$ have been assumed differentiable, integration by parts yields

$$\frac{dV_2(t,x_t)}{dt}\Big|_{x=x^u} = 2x'(t)A'(t)w_1(t)x(t)+2x'(t-h)B'(t)w_1(t)x(t)$$

$$+2u'(t)D'(t)w_1(t)x(t)+x'(t)\dot{w}_1(t)x(t)+2x'(t)A'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds$$

$$+2x'(t-h)B'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds+2u'(t)D'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds$$

$$+2x'(t)\int_{-h}^{T-h-t} \frac{\partial w_2(t,s)}{\partial t} x(t+s)ds+2x'(t)w_2(t,s)x(t+s)\Big|_{s=-h}^{s=T-h-t}$$

$$-2x'(t)\int_{-h}^{T-h-t} \frac{\partial w_2}{\partial s} x(t+s)ds-2x'(t)w_2(t,T-h-t)x(T-h)$$

$$+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t} x'(t+s) \frac{\partial w_3(t,r,s)}{\partial t} x(t+r)drs$$

$$+\int_{-h}^{T-h-t} [x'(t+s)w_3(t,r,s)x(t+r)\Big|_{r=-h}^{r=T-h-t}$$

$$-\int_{-h}^{T-h-t} x'(t+s) \frac{\partial w_3(t,r,s)}{\partial r} x(t+r)dr]ds$$

$$+\int_{-h}^{T-h-t}[x'(t+s)w_3(t,r,s)x(t+r)\Big|_{s=-h}^{s=T-h-t}$$

$$-\int_{-h}^{T-h-t} x'(t+s)\frac{\partial w_3(t,r,s)}{\partial s}x(t+r)ds]dr$$

$$-\int_{-h}^{T-h-t} x'(T-h)w_3(t,r,T-h-t)x(t+r)dr$$

$$-\int_{-h}^{T-h-t} x'(t+s)w_3(t,T-h-t,s)x(T-h)ds.$$

Adding $x'(t)Q(t)x(t)+u'(t)R(t)u(t)$ to both sides and grouping terms yields

$$\frac{dV_2}{dt}(t,x_t)\Big|_{x=x^u} + x'(t)Q(t)x(t)+u'(t)R(t)u(t) = x'(t)[2A'(t)w_1(t)$$

$$+Q(t)+\dot{w}_1(t)]x(t)+x'(t-h)[2B'(t)w_1(t)-2w_2'(t,-h)]x(t)$$

$$+x'(t)\int_{-h}^{T-h-t}\{-2\frac{\partial w_2(t,s)}{\partial s}+2\frac{\partial w_2(t,s)}{\partial t}+2A'(t)w_2(t,s)\}x(t+s)ds$$

$$+x'(t-h)\int_{-h}^{T-h-t}\{2B'(t)w_2(t,s)-2w_3'(t,-h,s)\}x(t+s)ds$$

$$+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t} x'(t+s)\{\frac{\partial w_3(t,r,s)}{\partial t}-\frac{\partial w_3(t,r,s)}{\partial s}-\frac{\partial w_3(t,r,s)}{\partial r}\}x(t+r)drds$$

$$+2u'(t)D'(t)w_1(t)x(t)+2u'(t)D'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds$$

$$+u'(t)R(t)u(t). \tag{26}$$

Now a u2(t) will be found such that (iii) of Lemma 1 is satisfied. u2(t) is found just as u1(t) was previously, and is a global minimum of (26);

$$u2(t) = -R^{-1}(t)D'(t)w_1(t)x(t)-R^{-1}(t)D'(t)\int_{-h}^{T-h-t} w_2(t,s)x(t+s)ds \tag{27}$$

This is substituted into (26) to obtain

$$\frac{dV_2(t,x_t)}{dt}\Big|_{x=x^u}+x'(t)Q(t)x(t)+u'(t)R(t)u(t)=x'(t)[2A'(t)w_1(t)+Q(t)$$

$$+\dot{w}_1(t)+w_1(t)D(t)R^{-1}(t)D'(t)w_1(t)]x(t)+2x'(t-h)[B'(t)w_1(t)$$

$$-w_2'(t,-h)]x(t)+x'(t)\int_{-h}^{T-h-t}[-2\frac{\partial w_2(t,s)}{\partial s}+2\frac{\partial w_2(t,s)}{\partial t}+2A'(t)w_2(t,s)$$

$$-2w_1(t)D(t)R^{-1}(t)D'(t)w_2(t,s)]x(t+s)ds+x'(t-h)\int_{-h}^{T-h-t}[2B'(t)w_2(t,s)$$

$$-2w_3'(t,-h,s)]x(t+s)ds+\int_{-h}^{T-h-t}\int_{-h}^{T-h-t}x'(t+s)[\frac{\partial w_3(t,r,s)}{\partial t}$$

$$-\frac{\partial w_3(t,r,s)}{\partial s}-\frac{\partial w_3(t,r,s)}{\partial r}-w_2'(t,s)D(t)R^{-1}(t)D'(t)w_2(t,r)]x(t+r)drds \qquad (28)$$

In order to satisfy condition (ii) of Lemma 1 Equation (28) is equated to zero for all $x_t \epsilon C^1[-h,0]$ and for all $T \geq t \geq T-h$. As before a necessary and sufficient condition for this to be true for $T \geq t \geq T-h$ is

$$\dot{w}_1(t)-w_1(t)D(t)R^{-1}(t)D'(t)w_1(t)+A'(t)w_1(t)+w_1(t)A(t)+Q(t)=0 \qquad (29)$$

$$\frac{\partial w_2(t,s)}{\partial t}-\frac{\partial w_2(t,s)}{\partial s}+A'(t)w_2(t,s)-w_1(t)D(t)R^{-1}(t)D'(t)w_2(t,s)=0 \qquad (30)$$

$$\frac{\partial w_3(t,r,s)}{\partial t}-\frac{\partial w_3(t,r,s)}{\partial s}-\frac{\partial w_3(t,r,s)}{\partial r}-w_2'(t,s)D(t)R^{-1}(t)D'(t)w_2(t,r)=0 \quad (31)$$

$$B'(t)w_1(t)=w_2'(t,-h) \qquad (32)$$

and

$$B'(t)w_2(t,s)=w_3'(t,-h,s) \qquad (33)$$

where $-h \leq s \leq T-h-t$, $-h \leq r \leq T-h-t$.

Since $V_1(t,x_t)$ is to be the optimal cost functional for $t < T-h$ and $V_2(t,x_t)$ is the optimal cost functional for $t \geq T-h$ it is clear that

$$V_1(T-h,x_{T-h})=V_2(T-h,x_{T-h})$$

or

$$x'(T-h)[p_1(T-h)-w_1(T-h)]x'(T-h)+x'(T-h)\int_{-h}^{0}[p_2(T-h,s)-w_2(T-h,s)]ds$$
$$+\int_{-h}^{0}\int_{-h}^{0}x'(T-h+s)[p_3(T-h,r,s)-w_3(T-h,r,s)]x(T-h+r)drds=0 \tag{34}$$

must be satisfied. Equation (34) is satisfied if and only if Corollary 1 and 2 Appendix 2)

$$p_1(T-h) = w_1(T-h) \tag{35}$$

$$p_2(T-h,s) = w_2(T-h,s) \tag{36}$$

$$p_3(T-h,r,s) = w_3(T-h,r,s) \tag{37}$$

To actually force (35)-(37) to be satisfied, $w_1(T-h)$, $w_2(T-h,s)$ and $w_3(T-h,r,s)$ may be used as additional boundary conditions for (16)-(18). $V_1(t,x_t)$ and $V_2(t,x_t)$ and $u1(t)$ and $u2(t)$ are now combined in the following way.

$$u0(t) = \begin{cases} u1(t) & \text{for } t < T-h \\ u2(t) & \text{for } t \geq T-h \end{cases} \tag{38}$$

$$V(t,x_t) = \begin{cases} V_1(t,x_t) & \text{for } t < T-h \\ V_2(t,x_t) & \text{for } t-h \leq t \leq T \end{cases} \tag{39}$$

It is seen that $u0(t)$ and $V(t,x_t)$ defined in this way satisfy conditions (ii) and (iii) of Lemma 1, all that remains is to satisfy condition (i) of Lemma 1. To do this

$$w_1(T) = F \tag{40}$$

Therefore, $V(t,x_t)$ and $u0(t)$ satisfy Lemma 1 and are hence the optimal cost and control respectively.

Equations (6) and (7) are now derived. These equations arise from a close examination of a delay system when τ, the initial time, is within a delay interval of the final time. If $\tau \geq T-h$, $x(t-h)$ becomes a known function of time over the entire optimization interval and is given as a part of the initial condition. From this it is clear that the optimization problem for the delay system (1) is equivalent to minimization of (2) subject to

$$\dot{x}(t) = A(t)x(t) + z(t) + D(t)u(t)$$

where

$$z(t) = B(t)x(t-h) \text{ and } \tau \geq T-h.$$

The optimal cost for the new problem depends only on $A(t)$, $z(t)$, $D(t)$, $R(t)$, $Q(t)$, F, T and $x(\tau)$ [7]. Once these functions are chosen the optimal cost is fixed. Since the two systems are equivalent, $V(\tau, x_\tau)$ for $\tau > T-h$, as expressed in (5), must be such that it does not depend on $x(t)$ for $T-h \leq t < \tau$. A necessary and sufficient condition [Appendix 2] for this to be true is that Equations (6) and (7) be satisfied.

IV. SOME OBSERVATIONS

The solution of the problem is complicated by the need to solve a system of partial differential equations. Several ideas will now be presented which will give more insight into methods of solution and utilization of the boundary conditions.

$V_1(t, x_t)$ in Equation (8) may have been selected as

$$V_1^*(t,x_t) = x'(t)p_1^*(t)x(t) + x'(t)\int_{t-h}^{t} p_2^*(t,q)x'(q)dq + \int_{t-h}^{t}\int_{t-h}^{t} x'(q)p_3^*(t,q,v)x(v)dqdv.$$

This results in a different but equivalent form of the equations involving the p^*'s. These equations may be derived by assumming

$$p_2(t,s) = p_2^*(t,t+s) \tag{43}$$

and

$$p_3(t,r,s) = p_3^*(t,t+r,t+s). \tag{44}$$

Substituting Equations (43) and (44) into (16)-(20) and noting that

$$\frac{\partial p_2^*(t,t+s)}{\partial t} = \frac{\partial p_2^*(t,q)}{\partial t}\Big|_{q=t+s} + \frac{\partial p_2^*(t,q)}{\partial q}\Big|_{q=t+s}$$

and

$$\frac{\partial p_3^*(t,t+r,t+s)}{\partial t} = \frac{\partial p_3^*(t,q,v)}{\partial t}\Big|_{\substack{q=t+s\\v=t+r}} + \frac{\partial p_3^*(t,q,v)}{\partial q}\Big|_{\substack{q=t+s\\v=t+r}} + \frac{\partial p_3^*(t,q,v)}{\partial v}\Big|_{\substack{q=t+s\\v=t+r}}$$

yields, for $t \leq T-h$, $t-h \leq q \leq t$, $t-h \leq v \leq t$

$$\dot{p}_1^*(t)+A'(t)p_1^*(t)+p_1^*(t)A(t)+p_2^*(t,t)+p_2^{*\prime}(t,t)-p_1^*(t)D(t)R^{-1}(t)D'(t)p_1^*(t)$$
$$+Q(t) = 0 \tag{45}$$

$$B'(t)p_1^*(t) = p_2^{*\prime}(t,t-h) \tag{46}$$

$$\frac{\partial p_2^*(t,q)}{\partial t}+A'(t)p_2^*(t,q)+p_3^*(t,q,t)-p_1^*(t)D(t)R^{-1}(t)D'(t)p_2^*(t,q) = 0 \tag{47}$$

$$B'(t)p_2^*(t,q) = p_3^*(t,q,t-h) \tag{48}$$

$$\frac{\partial p_3^*(t,q,v)}{\partial t} - p_2^{*\prime}(t,q)D(t)R^{-1}(t)D'(t)p_2^*(t,v) = 0 \tag{49}$$

The same procedure may be carried out for $V_2(t,x_t)$ where $t \geq T-h$ yielding

$$\dot{w}_1^*(t)+A'(t)w_1^*(t)+w_1^*(t)A(t)-w_1^*(t)D(t)R^{-1}(t)D'(t)w_1^*(t)+Q(t) = 0 \tag{50}$$

$$B'(t)w_1^*(t) = w_2^{*\prime}(t,t-h) \tag{51}$$

$$\frac{\partial w_2^*(t,q)}{\partial t}+A'(t)w_2^*(t,q)-w_1^*(t)D(t)R^{-1}(t)D'(t)w_2^*(t,q) = 0 \tag{52}$$

$$B'(t)w_2^*(t,q) = w_3^*(t,q,t-h) \tag{53}$$

$$\frac{\partial w_3^*(t,q,v)}{\partial t} - w_2^{*\prime}(t,q)D(t)R^{-1}(t)D'(t)w_2^*(t,v) = 0 \tag{54}$$

where $T \leq t \leq T-h$, $t-h \leq q \leq T-h$, $t-h \leq v \leq T-h$.

Substituting $p_2^*(t,t+s)$ into Equation (38) for the optimal control yields for $t \leq T-h$,

$$u1(t) = -R^{-1}(t)D'(t)p_1^*(t)x(t) - R^{-1}(t)D'(t)\int_{-h}^{0} p_2^*(t,t+s)x(t+s)ds$$

Let $q = t+s$ and $u1(t)$ becomes

$$u1(t) = -R^{-1}(t)D'(t)p_1^*(t)x(t) - R^{-1}(t)D'(t)\int_{t-h}^{t} p_2^*(t,q)x(q)dq \qquad (55)$$

Then for $T-h \leq t \leq T$

$$u2(t) = -R^{-1}(t)D'(t)w_1^*(t)x(t) - R^{-1}(t)D'(t)\int_{t-h}^{T-h} w_2^*(t,q)x(q)dq \qquad (56)$$

A technique for the solution of (50)-(54) will now be presented. First, $w_1^*(t)$ is obtained from (50), using $w_1^*(T) = F$. Utilizing $w_1^*(t)$ and Eq. (51), Figure 1 illustrates the information known about $w_2^*(t,q)$ as well as a possible approach to the solution of Equation (52). First (52) is solved for q fixed at q_1, $T-2h < q_1 \leq T-h$.

This solution will involve an unknown constant and will be valid along a line of $q=q_1$ for $T-h \leq t \leq q_1+h$ as illustrated in Figure 1. Then t is varied until a boundary condition can be used to evaluate the unknown constant. From Figure 1 it is seen that if $t=q_1+h$, the boundary condition (51) may be used to evaluate the unknown constant.

Carrying out the above idea analytically, the solution of Equation (52) is of the form

$$w_2^*(t,q_1) = \Phi(t,\tau)c_1$$

where $\Phi(t,\tau)$ is a solution of (52) such that $\Phi(\tau,\tau) = I$. If t is now changed and q_1 held constant, the boundary condition at $t=q_1+h$ may be used to evaluate c_1, yielding

$$w_2^*(t,q_1) = \Phi(t,q_1+h)w_1^*(q_1+h)B(q_1+h) \quad T-h \leq t \leq q_1+h \qquad (57)$$

Utilizing Equation (57) it is clear that $w_2^*(t,q)$ can be calculated for $T-h \leq t \leq T$, $t-h \leq q \leq T-h$. Knowing $w_2^*(t,q)$, $w_3^*(t,q,v)$ may also be determined by the above approach. Equation (54) is solved for q and v fixed at q_1 and v_1. The solution involves an unknown constant as before. Then t is varied along the line t, q_1, v_1 until boundary conditions (53) can be used to evaluate the unknown constant.

In Equation (57) if $F \neq 0$ and $q_1 = T-h$ then

$$w_1^*(q_1+h) = F, \text{ hence}$$

$$w_2^*(t, T-h) = \Phi(t, T-h)FB(T-h).$$

Therefore, if $B(T-h) \neq 0$

$$w_2^*(t, T-h) \neq 0 \quad T-h \leq t \leq T.$$

Since it will be shown that (6) is satisfied, i.e.

$$w_2^*(t,q) = 0 \text{ for } T-h < q \leq t \quad \forall t > T-h,$$

it is clear that $w_2^*(t,q)$ has a discontinuity at $q=T-h$. In a similar fashion it can be shown that $w_3^*(t,q,v)$ is discontinuous at $q=T-h$ and $v \leq T-h$ or $q \leq T-h$ and $v = T-h$.

The main problem remaining is that of finding a technique for the total solution of the partial differential equations. One of the most powerful methods adaptable to this problem is that of finite difference. This method could be used with Equations (16)-(20) and (29)-(33) or (45)-(54), whichever is more desirable for a particular problem. It would be possible to use Equation (47) and its equivalent for $w_3^*(t,q,v)$ to obtain the solution for $t \geq T-h$. Using the solution at $t=T-h$ as a boundary condition, the solution could then be obtained for $t < T-h$ using difference techniques.

It is interesting to compare the optimal control generated to minimize (2) for $t_0 \geq T-h$ subject to

$$\dot{x}(t) = A(t)x(t)+D(t)u(t)+z(t) \tag{58}$$

where

$$z(t) = B(t)x(t-h)$$

and the optimal control to minimize (2) subject to (1). Since, as was mentioned, the systems are equivalent, the controls should be equal. The optimal control for (58) is [7]

$$u_R(t) = -R^{-1}(t)D'(t)K(t)x(t)+R^{-1}(t)D'(t)g(t)$$

where K(t) is a solution to the Riccati equation and

$$\dot{g}(t) = [-A(t)+D(t)R^{-1}(t)D(t)K(t)]g(t)-K(t)z(t) \tag{59}$$

$$g(T) = 0.$$

Since

$$z(t) = B(t)x(t-h),$$

$$u_R(t) = -R^{-1}(t)D'(t)K(t)x(t)+R^{-1}(t)D'(t)\int_T^t \Phi_R(t,\sigma)K(\sigma)B(\sigma)x(\sigma-h)d\sigma$$

where $\Phi_R(t,\sigma)$ is the fundamental matrix for (59). Letting $\sigma=q+h$ and reversing the limits of integration yield

$$u_R(t)=-R^{-1}(t)D'(t)K(t)x(t)-R^{-1}(t)D'(t)\int_{t-h}^{T-h}\Phi_R(t,q+h)K(q+h)B(q+h)x(q)dq \tag{60}$$

Utilizing Equations (56) and (57), the optimal control for (1) is

$$u2(t)=-R^{-1}(t)D'(t)w_1^*(t)x(t)-R^{-1}(t)D'(t)\int_{t-h}^{T-h}\Phi(t,q_1+h)w_1^*(q+h)B(q_1+h)x(q_1)dq$$

Since $w_1^*(t) = K(t)$ and $\Phi_R(t,\sigma) = \Phi(t,\sigma)$ it is seen that $u_R(t)=u2(t)$.

A method has been presented which generates an optimal feedback controller for time varying systems with delay. A set of partial differential equations and boundary conditions, whose solution yields an optimal feedback controller, are presented. The existence of a solution to these equations has not been proven and would be a problem for further research. Techniques for the possible numerical solution have been suggested, although their convergence for this problem has not been investigated. For the degenerate case where $h=0$ the above results reduce to those obtained previously for non-time delay systems [7].

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APPENDIX I

It will be shown that no loss of generality results when $p_1(t)$ and $p_3(t,r,s)$ are restricted by

$$p_1(t) = p_1'(t)$$

$$p_3(t,r,s) = p_3'(t,s,r).$$

Lemma 1.

$$\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\left[\frac{G(t,r,s)+G'(t,s,r)}{2}\right]x(t+s)\,dr\,ds$$

$$= \int_{-h}^{0}\int_{-h}^{0} x'(t+r)G(t,r,s)x(t+s)\,dr\,ds$$

for all continuous $G(t,r,s)$.

Proof.

$$\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{[G(t,r,s)+G'(t,s,r)]}{2}\,x(t+s)\,dr\,ds$$

$$=\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{G(t,r,s)}{2}x(t+s)\,dr\,ds+\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{G'(t,s,r)}{2}x(t+s)\,dr\,ds.$$

Transposing the second term on the right hand side yields

$$\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{[G(t,r,s)+G'(t,s,r)]}{2}x(t+s)\,dr\,ds$$

$$=\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{G(t,r,s)}{2}x(t+r)\,dr\,ds+\int_{-h}^{0}\int_{-h}^{0} x'(t+s)\frac{G(t,s,r)}{2}x(t+r)\,dr\,ds.$$

Since r and s are only dummy variables and have the same limits of integration, interchange r and s, and then the order of integration yielding

$$\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{[G(t,r,s)+G'(t,s,r)]}{2}\,x(t+s)\,dr\,ds$$

$$=\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{G(t,r,s)}{2}x(t+s)\,dr\,ds+\int_{-h}^{0}\int_{-h}^{0} x'(t+r)\frac{G(t,r,s)}{2}x(t+s)\,dr\,ds$$

$$=\int_{-h}^{0}\int_{-h}^{0} x'(t+r)G(t,r,s)x(t+s)\,dr\,ds$$

Q.E.D.

It is well known that

$$x'Fx = \frac{x'(F + F')x}{2} \quad \forall F.$$

Hence the form of $p_1(t)$ and $p_3(t,r,s)$ may be restricted and still generate the same functionals obtainable by arbitrary $p_1(t)$ and $p_3(t,r,s)$.

APPENDIX 2

Theorem 1.

$V(\tau, x_\tau)$ for $T \geq \tau > T-h$ does not depend on $x(t)$ for $T-h < t < \tau$ if and only if

$$p_2(t,s) = 0 \qquad t \geq s+t > T-h \tag{A.1}$$

$$p_3(t,r,s) = 0 \qquad t \geq s+t > T-h \text{ or } t > r+t > T-h \tag{A.2}$$

where $p_2(t,s)$ is continuous for $T-h < t+s \leq t$ and $p_3(t,r,s)$ is continuous for $t \geq s+t > T-h$ or $t \geq r+t > T-h$.

Proof.

For $T > \tau > T-h$

$$\begin{aligned}
V(\tau, x_\tau) = {} & x'(\tau)p_1(\tau)x(\tau) + x'(\tau)\int_{-h}^{T-h-\tau} p_2(\tau,s)x(\tau+s)ds \\
& + \int_{-h}^{T-h-\tau}\int_{-h}^{T-h-\tau} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds + x'(\tau)\int_{T-h-\tau}^{0} p_2(\tau,s)x(\tau+s)ds \\
& + \int_{T-h-\tau}^{0}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds \\
& + \int_{-h}^{T-h-\tau}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds \\
& + \int_{T-h-\tau}^{0}\int_{-h}^{T-h-\tau} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds
\end{aligned} \tag{A.3}$$

It is clear that only the last four terms of (A.3) depend on $x(t)$ for $T-h < t < \tau$ and that if $x(t) = 0$ for $T-h < t < \tau$ the last four terms of (A.3) are zero; hence if they are to be independent of $x(t)$ for $T-h < t < \tau$ they must always be zero. A sufficient condition for this is that Equation (A.1) and (A.2) be satisfied. In order to establish necessity two corollaries will now be proved.

Corollary 1.

$$B(a,b)=\int_a^b\int_a^b x'(r)p(r,s)x(s)drds=0 \quad \forall\ x\in C[a,b] \ni x(a)=x(b)=0 \qquad (A.4)$$

implies $p(r,s) = 0$ for $a < r \leq b$ and $a < s \leq b$, where $p(r,s)$ is an nxn matrix, $x(t)$ is an n vector,

$p(r,s) = p'(s,r)$ and $p(r,s)$ is continuous for $a < r \leq b$ and $a < s \leq b$.

Proof.

Let $x_k = 0,\ \forall\ k \neq i$

then

$$B(a,b) = \int_a^b\int_a^b x_i(t)p_{ii}(r,s)x_i(s)drds \qquad (A.5)$$

assume $p_{ii}(q,q) > 0$ where $q\in(a,b)$ then since $p(r,s)$ is continuous $\exists\ \varepsilon > 0 \ni p_{ii}(r,s) > 0$ for $q-\varepsilon \leq r \leq q+\varepsilon$ and $q-\varepsilon \leq s \leq q+\varepsilon$. Let

$x_i(r) > 0$ for $a < q-\varepsilon < r < q+\varepsilon < b$ and $x_i(r) = 0$ elsewhere where $x_i(r)$ is continuous then

$$B(a,b) = \int_{q-\varepsilon}^{q+\varepsilon}\int_{q-\varepsilon}^{q+\varepsilon} x_i(r)p_{ii}(r,s)x_i(s)drds \qquad (A.6)$$

since $x(r)p_{ii}(r,s)x(s) > 0$ for $q-\varepsilon < r < q+\varepsilon$ and $q-\varepsilon < s < q+\varepsilon$ $B(a,b) > 0$. This may be repeated for $p_{ii}(q,q) < 0$ and obtain $B(a,b) < 0$. These are both contradictions, therefore

$p_{ii}(q,q) = 0$ for $q\in(a,b)$

Now assume $p_{ii}(r_1,s_1) > 0$ for $r_1\in(a,b)$ and $s_1\in(a,b)$. Then $\exists\ \varepsilon_1 > 0 \ni p(r,s) > 0$ for $r_1-\varepsilon_1 \leq r \leq r_1+\varepsilon_1$ and $s_1-\varepsilon_1 \leq s \leq s_1+\varepsilon_1$. Define $(r,s)\in S$ iff $r_1-\varepsilon_1 \leq r \leq r_1-\varepsilon_1$ and $s_1-\varepsilon_1 \leq s \leq s_1-\varepsilon_1$. Let

$$z(\varepsilon_1) = \inf_S p(r,s)$$

Note that $z(\epsilon_1) > 0$ and $z(\epsilon_1) \leq z(\epsilon_2)$ where $\epsilon_1 \geq \epsilon_2$. Since $p(r,s)$ is continuous and $p(q,q) = 0 \,\forall q \epsilon (a,b)$, given $\epsilon > 0 \; \exists \, \delta > 0$

$\ni \quad p(r,s) < \epsilon$ for $q-\delta \leq r \leq q+\delta$ and $q-\delta \leq s \leq q+\delta \;\forall q \epsilon (a,b)$.

Let $\epsilon = \dfrac{z(\epsilon_1)}{4}$, and find corresponding $\delta(\epsilon_1)$. Let $\epsilon_2 = \inf(\delta(\epsilon_1), \epsilon_1)$.

Let $x_i(r) = 1$ for $r_1 - \dfrac{\epsilon_2}{2} \leq r \leq r_1 + \dfrac{\epsilon_2}{2}$ and $1 > x_i(r) > 0$ for $a < r_1 - \epsilon_2 < r < r_1 + \epsilon_2 < b$ or $a < s_1 - \epsilon_2 < r < s_1 + \epsilon_2 < b$ and $x_i(r) = 0$ elsewhere. Also, $x_i(r)$ is continuous, therefore

$$
\begin{aligned}
B(a,b) = & \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds \\
& + \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds \\
& + \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} \int_{r_1-\epsilon_2}^{r_1-\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds \\
& + \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds.
\end{aligned}
\tag{A.7}
$$

but since $p(r,s) = p'(s,r)$ the last two terms of Equation (A.7) are equal.

Therefore

$$
\begin{aligned}
B(a,b) = & \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds \\
& + \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds \\
& + 2\int_{s_1-\epsilon_2}^{s_1+\epsilon_2} \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} x_i(r) p_{ii}(r,s) x_i(s)\, dr\, ds
\end{aligned}
\tag{A.8}
$$

Since $x_i(r)p_{ii}(r,s)x(s) > 0$ for $r_1-\epsilon_2 < r < r_1+\epsilon_2$ and $s_1-\epsilon_2 < s < s_1+\epsilon_2$ if the area of integration is reduced in the last integral of Equation (A.8) the value of the integral will be reduced. Therefore

$$B(a,b) > \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} \int_{r_1-\epsilon_2}^{r_1+\epsilon_2} x_i(r)p_{ii}(r,s)x_i(s)drds$$

$$+\int_{s_1-\epsilon_2}^{s_1+\epsilon_2} \int_{s_1-\epsilon_2}^{s_1+\epsilon_2} x_i(r)p_{ii}(r,s)x_i(s)drds$$

$$+2\int_{s_1-\epsilon_2/2}^{s_1+\epsilon_2/2} \int_{r_1-\epsilon_2/2}^{r_1+\epsilon_2/2} x_i(r)p_{ii}(r,s)x_i(s)drds \qquad \text{(A.9)}$$

Since $x_i(s)p_{ii}(r,s)x_i(r) > z(\epsilon_1)$ for $r_1-\epsilon_2 \leq r \leq r_1-\epsilon_2$ and $s_1-\epsilon_2 \leq s \leq s_1+\epsilon_2$ and $x_i(r)p_{ii}(r,s)x_i(s) < z(\epsilon_1)/4$ for $r_1-\epsilon_2 \leq r \leq \epsilon_2+r_1$ and $r_1-\epsilon_2 \leq s \leq \epsilon_2+r_1$ and $x_i(r)p_{ii}(r,s)x_i(s) < z(\epsilon_1)/4$ for $s_1-\epsilon_2 \leq r \leq s_1+\epsilon_2$ and $s_1-\epsilon_2 \leq s \leq s_1+\epsilon_2$. Equation (A.9) reduces to

$$B(a,b) > -2\,\frac{z(\epsilon_1)}{4}\,(2\epsilon_2)^2+2\;z(\epsilon_1)\epsilon_2^2 = 0$$

Therefore $B(a,b) > 0$.

The same result can be obtained if $p_{ii}(r_1, s_1)$ is assumed less than zero. These are both contradictions, therefore

$$p_{ii}(r,s) = 0 \;\forall a < r < b \text{ and } \forall a < s < b.$$

Now let $x_k = 0 \;\forall k \neq i,j \quad i \neq j$

Therefore

$$B(a,b) = \int_a^b \int_a^b x_i(r)p_{ij}(r,s)x_j(s)drds$$

$$+\int_a^b \int_a^b x_j(r)p_{ji}(r,s)x_i(s)drds \qquad \text{(A.10)}$$

Since

$$p(r,s) = p'(s,r)$$

$$p_{ij}(r,s) = p_{ji}(s,r)$$

Therefore

$$B(a,b) = \int_a^b \int_a^b x_i(r)p_{ij}(r,s)x_j(s)drds$$

$$+ \int_a^b \int_a^b x_j(r)p_{ij}(s,r)x_i(s)drds \tag{A.11}$$

Since limits are the same on both variables the terms in Equation (A.11) are equal and

$$B(a,b) = 2\int_a^b \int_a^b x_i(r)p_{ij}(r,s)x_j(s)drds \tag{A.12}$$

assume $p_{ij}(r_1, s_1) > 0$ where $r_1 \epsilon (a,b)$, $s_1 \epsilon (a,b)$. Then $\exists \epsilon > 0 \ni p_{ij}(r,s) > 0$ for $a < r_1 - \epsilon \leq r \leq r_1 + \epsilon < b$ and $a < s_1 + \epsilon \leq s \leq s_1 + \epsilon < b$.

Let $x_j(r) > 0$ for $r_1 - \epsilon < r < r_1 + \epsilon$, $x_j(r) = 0$ elsewhere, $x_i(s) > 0$ for $s_1 - \epsilon < s < s_1 + \epsilon$, $x_i(s) = 0$ elsewhere and $x_i(r)$ and $x_i(s)$ are continuous. Therefore

$$B(a,b) = 2\int_{r_1-\epsilon}^{r_1+\epsilon} \int_{s_1-\epsilon}^{s_1+\epsilon} x_i(r)p_{ij}(r,s)x_j(r)drds$$

since $x_j(r)p_{ij}(r,s)x_i(s) > 0$ for $r_1 - \epsilon < r < r_1 + \epsilon$ and $s_1 - \epsilon < s < s_1 + \epsilon$, $B(a,b) > 0$. If $p_{ij}(r_1, s_1)$ had been assumed less than zero $B(a,b)$ would have been less than zero. Both results are contradictions, therefore

$p_{ij}(r,s) = 0$ for $a < r < b$ and $a < s < b$.

It has been shown that $p(r,s) = 0$ for $a < r < b$ and $a < s < b$. From continuity $p(r,s) = 0$ for $a < r \leq b$ and $a < s \leq b$.

Q.E.D.

Corollary 2.

$$C(a,b) = x'(b)\int_a^b p(s)x(s)ds = 0 \ \forall x \epsilon C[a,b]$$

implies $p(s) = 0$ for $a < s \leq b$, where $p(s)$ is an nxn matrix continuous for $a < s \leq b$.

Proof.

Let $x_k = 0 \ k \neq i$ then

$$C(a,b) = x_i(b)\int_a^b p_{ii}(s)x_i(s)ds$$

assume $p_{ii}(b) > 0$, let $x_i(s) = p_{ii}(s)$ for $a < s \leq b$, since $p_{ii}(s)$ is continuous $\exists$ an $\epsilon > 0 \ni p_{ii}(s) > 0$ for $b \geq s \geq b-\epsilon$. Therefore

$$C(a,b) > p_{ii}(b)\int_{b-\epsilon}^b p_{ii}^2(s)ds > 0$$

and for $p_{ii}(b) < 0$ it can be shown that $C(a,b) < 0$. These are both contradictions, therefore $p_{ii}(b) = 0$.

Choose $x_i(s) = p_{ii}(s)$ $a < s \leq b-\epsilon$ $\epsilon > 0$, $x_i(b) = 1$ and $x_i(s)$ continuous, therefore

$$C(a,b) = \int_a^{b-\epsilon} p_{ii}^2(s)ds + \int_{b-\epsilon}^b p_{ii}(s)x_i(s)ds.$$

Now if $p_{ii}(s_1) > 0$ for $s_1 < b$ choose ϵ such that $b-\epsilon > s_1$ then

$$\int_a^{b-\epsilon} p_{ii}^2(s)ds \geq \int_a^{b-\epsilon_1} p_{ii}^2(s)ds > 0 \text{ where } \epsilon < \epsilon_1 \text{ and}$$

$\int_{b-\epsilon}^b p_{ii}(s)x_i(s)ds$ can be made arbitrarily small by decreasing ϵ, therefore $C(a,b) > 0$.

For $p_{ii}(s_1) < 0$ it can be shown that $C(a,b) < 0$. These are both contradictions, therefore

$$p_{ii}(s) = 0 \quad a < s \leq b.$$

Now let $x_k = 0 \quad k \neq i, j \qquad i \neq j$, then

$$C(a,b) = x_i(b) \int_a^b p_{ij}(s)x_j(s) + x_j(b) \int_a^b p_{ij}(s)x_i(s)ds.$$

Assume

$$p_{ij}(s_1) > 0 \quad s_1 < b.$$

Then $\exists \, \epsilon > 0 \ni p_{ij}(s) > 0$ for $s_1 - \epsilon \leq s \leq s_1 + \epsilon$ and $s_1 + \epsilon < b$.

Let $x_j(b) = 0$, $x_i(b) = 1$ and $x_j(s) > 0$ for $s_1 - \epsilon < s < s_1 + \epsilon$, $x_j(s) = 0$ elsewhere.

$$C(a,b) = \int_{s_1-\epsilon}^{s_1+\epsilon} p_{ij}(s)x_j(s)ds > 0.$$

For $p_{ij}(s_1) < 0$ it can be shown that $C(a,b) < 0$. These are both contradictions, therefore $p_{ij}(s) = 0 \quad a < s < b$. Again by continuity

$p_{ij}(s) = 0 \; a < s \leq b$. Q.E.D.

It must now be shown that

$$x(\tau)\int_{T-h-\tau}^{0} p_2(\tau, s)x(\tau+s)ds + \int_{T-h-\tau}^{0} \int_{T-h-\tau}^{0} x'(\tau+r)p_3(\tau, r, s)x(\tau+s)drds$$

$$+ \int_{-h}^{T-h-\tau} \int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau, r, s)x(\tau+r)drds$$

$$+ \int_{T-h-\tau}^{0} \int_{-h}^{T-h-\tau} x'(\tau+s)p_3(\tau, r, s)x(\tau+r)drds = 0 \tag{A.13}$$

implies that Equations (A.1) and (A.2) are satisfied. First Equation (A.13) is simplified by noting that

$$p_3(\tau, r, s) = p_3'(\tau, s, r) \tag{A.14}$$

implies

$$\int_{T-h-\tau}^{0} \int_{-h}^{T-h-\tau} x'(\tau+s)p_3(\tau, r, s)x(\tau+r)drds$$

$$= \int_{T-h-\tau}^{0} \int_{-h}^{T-h-\tau} x'(\tau+r)p_3(\tau, s, r)x(\tau+s)drds \tag{A.15}$$

Interchanging the order of integration yields

$$\int_{T-h-\tau}^{0}\int_{-h}^{T-h-\tau} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds$$

$$=\int_{-h}^{T-h-\tau}\int_{T-h-\tau}^{0} x^T(\tau+r)p_3(\tau,s,r)x(\tau+s)dsdr \tag{A.16}$$

Since r and s are dummy variables they may be interchanged yielding

$$\int_{T-h-\tau}^{0}\int_{-h}^{T-h-\tau} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds$$

$$=\int_{-h}^{T-h-\tau}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds \tag{A.17}$$

Utilizing (A.17) Equation (A.13) becomes

$$x'(\tau)\int_{T-h-\tau}^{0} p_2(\tau,s)x(\tau+s)ds+\int_{T-h-\tau}^{0}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds$$

$$+2\int_{-h}^{T-h-\tau}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds=0 \tag{A.18}$$

Since $x(t)$ is arbitrary for $\tau-h\le t\le\tau$, let $x(\tau+q)=0$ for $-h\le q\le T-h-\tau$

and $x(\tau)=0$. Equation (A.18) becomes

$$\int_{T-h-\tau}^{0}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds=0 \tag{A.19}$$

Let $r+\tau = q$ and $s+\tau = v$, then (A.19) becomes

$$\int_{T-h}^{\tau}\int_{T-h}^{\tau} x'(q)p_3(\tau,q-\tau,v-\tau)x(v)dqdv=0 \quad \forall x\in C[T-h,0],\ x(T-h)=x(\tau)=0 \tag{A.20}$$

From Corollary 1

$$p_3(\tau,q-\tau,v-\tau)=0 \text{ for } T-h<q\le\tau \text{ and } T-h<v\le\tau \text{ or}$$

$$p_3(\tau,r,s)=0 \text{ for } T-h<\tau+r\le\tau \text{ and } T-h<\tau+s\le\tau. \tag{A.21}$$

Making use of (A.21) and unspecified $x(t)$ for $\tau-h\le t<\tau$ and $x(\tau)=0$

Equation (A.18) becomes

$$2\int_{-h}^{T-h-\tau}\int_{T-h-\tau}^{0} x'(\tau+s)p_3(\tau,r,s)x(\tau+r)drds=0 \tag{A.22}$$

assume $p_{3ij}(\tau, r_1, s_1) > 0$ for $-h < r_1 < T-h-\tau$ and $T-h-\tau < s_1 < 0$. Then $\exists$ an $\epsilon > 0 \ni p_{3ij}(\tau, r, s) > 0$ for $r_1-\epsilon \leq r \leq r_1+\epsilon$ and $s_1-\epsilon \leq s \leq s_1+\epsilon$ where $r_1+\epsilon < T-h-\tau < r_1-\epsilon$. Let $x_i(\tau+q) > 0$ for $r_1-\epsilon < q < r_1+\epsilon$, $x_j(\tau+q) > 0$ for $s_1-\epsilon < q < s_1+\epsilon$, $x_i(q) = 0$ elsewhere, $x_j(q) = 0$ elsewhere and $x_k = 0 \quad k \neq i, j$. Equation (A.22) becomes

$$2 \int_{s_1-\epsilon}^{s_1+\epsilon} \int_{r_1-\epsilon}^{r_1+\epsilon} x_j(\tau+s) p_{3ij}(\tau, r, s) x_i(\tau+r) dr ds = 0 \qquad \text{(A.23)}$$

but since $x_j(\tau+s) p_{3ij}(\tau, r, s) x_i(\tau+r) > 0$ for $r_1-\epsilon < r < r_1+\epsilon$ and $s_1-\epsilon < s < s_1+\epsilon$ Equation (A.23) is a contradiction. If $p_{ij}(\tau, r_1, s_1)$ had been assumed less than zero a contradiction would still have been obtained, therefore, $p_{3ji}(\tau, r, s) = 0$ for $T-h-\tau < s < 0$ and $-h < r < T-h-\tau$. As before continuity yields

$$p_{3ij}(\tau, r, s) = 0 \text{ for } T-h-\tau < s < 0 \text{ and } -h \leq r \leq T-h-\tau. \qquad \text{(A.24)}$$

Therefore, from (A.24) and (A.21) $p_{3ij}(\tau, r, s) = 0$ for $T-h < s+\tau \leq \tau$ and since $p_3(\tau, r, s) = p_3(\tau, s, r)$ then $p_{3ij}(\tau, r, s) = 0$ for $T-h < r+\tau \leq \tau$.

Rewriting Equation (A.13) yields $x'(\tau) \int_{T-h-\tau}^{0} p_2(\tau, s) x(\tau+s) ds = 0$ $\forall x_\tau \epsilon C[\tau-h, \tau]$. Letting $q = \tau+s$ yields

$$x'(\tau) \int_{T-h}^{\tau} p_2(\tau, q-\tau) x(q) dq = 0$$

From Corollary 2

$$p_2(\tau, q-\tau) = 0 \text{ for } T-h < q \leq \tau$$

or

$$p_2(\tau, s) = 0 \text{ for } T-h < s+\tau \leq \tau \quad . \qquad \text{Q.E.D.}$$

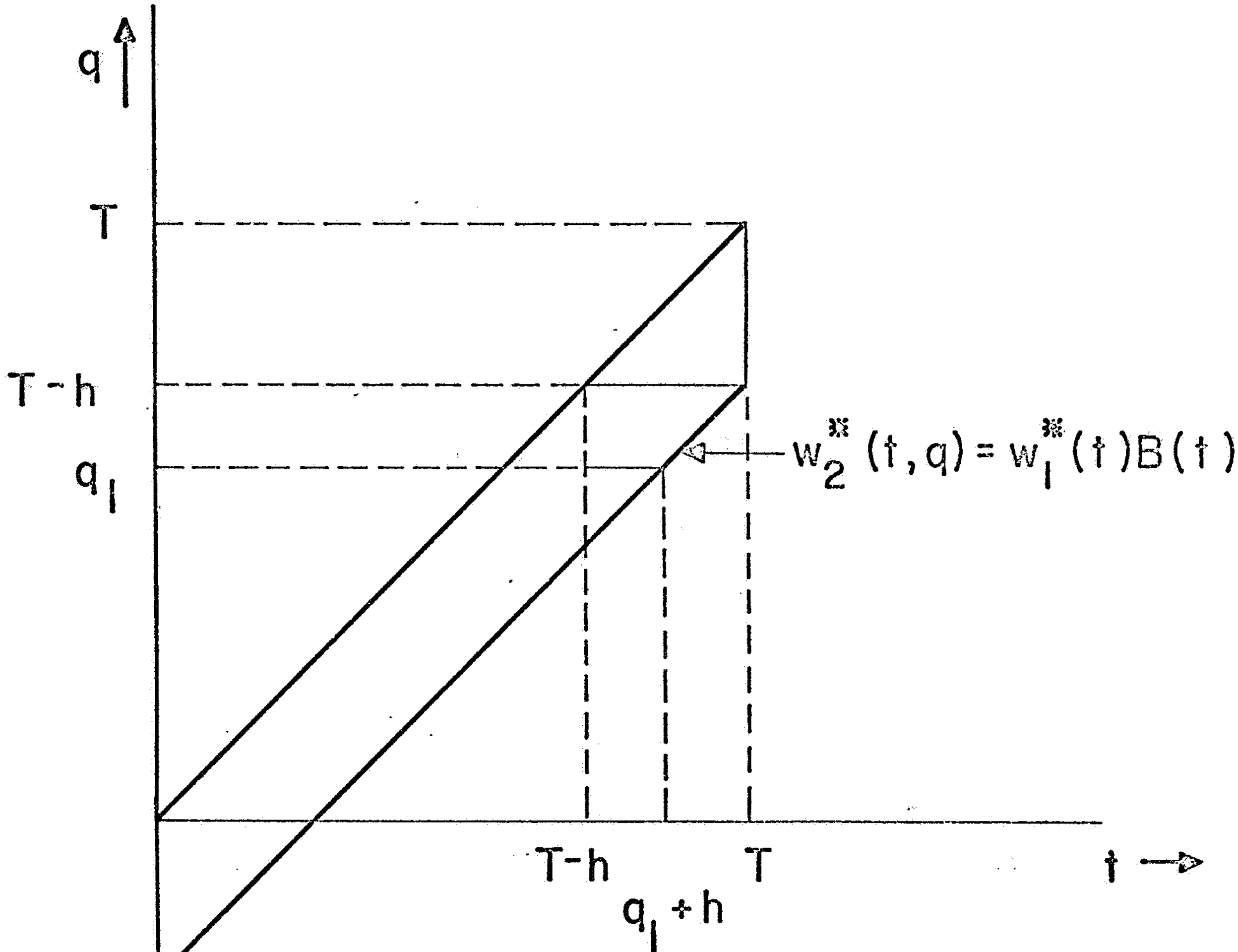

FIGURE 1